MTH 166

Lecture-25

Parametric Equation of a Tangent Line, Length of Space Curve, Motion of a Particle

Topic:

Vector Differential Calculus

Learning Outcomes:

1. To find parametric equation of tangent line

2. To find Length of Space Curve

3. Motion of a body or a particle

Parametric Equation of Tangent Line:

Problem 1. Find the parametric equation of tangent line to the curve:

$x = t, y = 2t^2, z = 3t^3$ at $t = 2$.

Solution. Let $P_0 = (x, y, z)_{t=2} = (t, 2t^2, 3t^3)_{t=2} = (2,8,24)$

Let $\vec{v} = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right)_{t=2} = \left(\hat{\imath} + 4t\hat{\jmath} + 9t^2\hat{k}\right)_{t=2} = (\hat{\imath} + 8\hat{\jmath} + 36\hat{k})$

Parametric equation of tangent line to the given curve is:

$\overrightarrow{r(t)} = \overrightarrow{P_0} + \mu\vec{v}$ where μ is parameter

$\Rightarrow \overrightarrow{r(t)} = (2\hat{\imath} + 8\hat{\jmath} + 24\hat{k}) + \mu(\hat{\imath} + 8\hat{\jmath} + 36\hat{k})$

$\Rightarrow \overrightarrow{r(t)} = (2 + \mu)\hat{\imath} + (8 + 8\mu)\hat{\jmath} + (24 + 36\mu)\hat{k}$ **Answer.**

Problem 2. Find the parametric equation of tangent line to the curve:

$x = t, y = e^t, z = 1$ at $t = 1$.

Solution. Let $P_0 = (x, y, z)_{t=2} = (t, e^t, 1)_{t=1} = (1, e, 1)$

Let $\vec{v} = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right)_{t=1} = \left(\hat{\imath} + e^t\hat{\jmath} + 0\hat{k}\right)_{t=1} = (\hat{\imath} + e\hat{\jmath})$

Parametric equation of tangent line to the given curve is:

$\overrightarrow{r(t)} = \overrightarrow{P_0} + \mu\vec{v}$ where μ is parameter

$\Rightarrow \overrightarrow{r(t)} = (\hat{\imath} + e\hat{\jmath} + \hat{k}) + \mu(\hat{\imath} + e\hat{\jmath})$

$\Rightarrow \overrightarrow{r(t)} = (1 + \mu)\hat{\imath} + (e + e\mu)\hat{\jmath} + \hat{k}$ **Answer.**

Length of Space Curve:

Let the curve C be represented in the parametric form as:

$$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \quad a \le t \le b$$

Then, length of curve C is given by:

$$l = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Problem 1. Find the length of the following curve:

$\overrightarrow{r(t)} = (a\cos t)\hat{\imath} + (a\sin t)\hat{\jmath}, \quad 0 \le t \le 2\pi$

Solution. Comparing with: $\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath}, \quad a \le t \le b$

$x(t) = a\cos t \quad \Rightarrow \frac{dx}{dt} = -a\sin t$ and $y(t) = a\sin t \quad \Rightarrow \frac{dy}{dt} = a\cos t$

Length of curve C is given by: $l = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\Rightarrow l = \int_{t=0}^{t=2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt = a\int_{t=0}^{t=2\pi} dt = a(2\pi - 0)$

$\Rightarrow l = 2\pi a$ units **Answer.**

Problem 2. Find the length of the following curve:

$$\overrightarrow{r(t)} = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath} + (3t)\hat{k}, \quad -2\pi \le t \le 2\pi$$

Solution. Comparing with: $\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \quad a \le t \le b$

$$x(t) = a\cos \Rightarrow \frac{dx}{dt} = -a\sin t,\ y(t) = a\sin t \Rightarrow \frac{dy}{dt} = a\cos t, z(t) = 3t \Rightarrow \frac{dz}{dt} = 3$$

Length of curve C is given by: $l = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$$\Rightarrow l = \int_{t=-2\pi}^{t=2\pi} \sqrt{(\sin t)^2 + (a\cos t)^2 + (3)^2}\,dt = \int_{t=-2\pi}^{t=2\pi} \sqrt{1+9}\,dt$$

$\Rightarrow l = \sqrt{10}(2\pi + 2\pi) = 4\pi\sqrt{10}$ units **Answer.**

Motion of a body or a particle

Let position vector of a particle be given by:

$$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

Then, we can calculate following quantities:

1. Velocity, $\overrightarrow{V(t)} = \frac{d}{dt}\left(\overrightarrow{r(t)}\right)$

2. Speed, $S = \left|\overrightarrow{V(t)}\right|$ (Magnitude of vector $\overrightarrow{V(t)}$)

3. Acceleration, $\overrightarrow{a(t)} = \frac{d}{dt}\left(\overrightarrow{V(t)}\right)$

Problem 1. The position vector of a moving particle is given by:

$\overrightarrow{r(t)} = (\cos t + \sin t)\hat{\imath} + (\sin t - \cos t)\hat{\jmath} + t\hat{k}$. Determine the velocity, speed and acceleration of the particle in the direction of the motion.

Solution. $\overrightarrow{r(t)} = (\cos t + \sin t)\hat{\imath} + (\sin t - \cos t)\hat{\jmath} + t\hat{k}$

Velocity, $\overrightarrow{V(t)} = \frac{d}{dt}\left(\overrightarrow{r(t)}\right) = \frac{d}{dt}\left[(\cos t + \sin t)\hat{\imath} + (\sin t - \cos t)\hat{\jmath} + t\hat{k}\right]$

$\Rightarrow \overrightarrow{V(t)} = (-\sin t + \cos t)\hat{\imath} + (\cos t + \sin t)\hat{\jmath} + 1\hat{k}$

Speed, $S = \left|\overrightarrow{V(t)}\right| = \sqrt{(-\sin t + \cos t)^2 + (\cos t + \sin t)^2 + (1)^2}$

$\Rightarrow S = \sqrt{2(sin^2 t + cos^2 t) + 1} = \sqrt{3}$

Acceleration, $\overrightarrow{a(t)} = \frac{d}{dt}\left(\overrightarrow{V(t)}\right) = \frac{d}{dt}\left[(-\sin t + \cos t)\hat{\imath} + (\cos t + \sin t)\hat{\jmath} + 1\hat{k}\right]$

$\Rightarrow \overrightarrow{a(t)} = (-\cos t - \sin t)\hat{\imath} + (-\sin t + \cos t)\hat{\jmath} + 0\hat{k}$

Problem 2. The position vector of a moving particle is given by:

$\overrightarrow{r(t)} = t^3\hat{\imath} + t\hat{\jmath} + t^2\hat{k}$. Determine the velocity, speed and acceleration of the particle in the direction of the motion.

Solution. $\overrightarrow{r(t)} = t^3\hat{\imath} + t\hat{\jmath} + t^2\hat{k}$

Velocity, $\overrightarrow{V(t)} = \frac{d}{dt}\left(\overrightarrow{r(t)}\right) = \frac{d}{dt}\left[t^3\hat{\imath} + t\hat{\jmath} + t^2\hat{k}\right]$

$\Rightarrow \overrightarrow{V(t)} = 3t^2\hat{\imath} + \hat{\jmath} + 2t\hat{k}$

Speed, $S = \left|\overrightarrow{V(t)}\right| = \sqrt{(3t^2)^2 + (1)^2 + (2t)^2}$

$\Rightarrow S = \sqrt{9t^4 + 4t^2 + 1}$

Acceleration, $\overrightarrow{a(t)} = \frac{d}{dt}\left(\overrightarrow{V(t)}\right) = \frac{d}{dt}\left[3t^2\hat{\imath} + \hat{\jmath} + 2t\hat{k}\right]$

$\Rightarrow \overrightarrow{a(t)} = 6t\hat{\imath} + 0\hat{\jmath} + 2\hat{k}$

<u>Rules of Vector Differentiation</u>

If $\overrightarrow{u(t)}, \overrightarrow{v(t)}$ are vector functions and α, β are constants, then following rules of differentiation hold for vector functions in same way as that for scalar function.

1. $\frac{d}{dt}[\alpha\vec{u} \pm \beta\vec{v}] = \alpha\frac{d}{dt}(\vec{u}) + \beta\frac{d}{dt}(\vec{v})$

2. $\frac{d}{dt}[\vec{u}.\vec{v}] = \vec{u}\frac{d}{dt}(\vec{v}) + \vec{v}\frac{d}{dt}(\vec{u})$ $\left(\frac{d}{dt}(Dot\ product\ between\ \vec{u}\ and \vec{v})\right)$

3. $\frac{d}{dt}[\vec{u} \times \vec{v}] = \vec{u} \times \frac{d}{dt}(\vec{v}) + \vec{v} \times \frac{d}{dt}(\vec{u})$ $\left(\frac{d}{dt}(Cross\ product\ between\ \vec{u}\ and \vec{v})\right)$

4. $\frac{d}{dt}\left[\frac{\vec{u}}{\vec{v}}\right] = \frac{\vec{v}\frac{d}{dt}(\vec{u}) - \vec{u}\frac{d}{dt}(\vec{v})}{(\vec{v})^2}$, provided $\vec{v} \neq 0$

Problem 1. If $\overrightarrow{u(t)} = (\sin 2t)\hat{\imath} - (\cos 2t)\hat{\jmath} + t\hat{k}$, $\overrightarrow{v(t)} = (\cos 2t)\hat{\imath} - (\sin 2t)\hat{\jmath} + t^2\hat{k}$

Find $\left[\overrightarrow{u(t)}.\overrightarrow{v(t)}\right]'$

Solution. Here we will first find dot product between $\overrightarrow{u(t)}$ and $\overrightarrow{v(t)}$

i.e. $\left[\overrightarrow{u(t)}.\overrightarrow{v(t)}\right] = \left[(\sin 2t)\hat{\imath} - (\cos 2t)\hat{\jmath} + t\hat{k}\right].\left[(\cos 2t)\hat{\imath} - (\sin 2t)\hat{\jmath} + t^2\hat{k}\right]$

$$= (\sin 2t)(\cos 2t) + (-\cos 2t)(-\sin 2t) + (t)(t^2)$$

$$= 2\sin 4t + t^3$$

So, $\left[\overrightarrow{u(t)}.\overrightarrow{v(t)}\right]' = \frac{d}{dt}\left[\overrightarrow{u(t)}.\overrightarrow{v(t)}\right] = \frac{d}{dt}[2\sin 4t + t^3]$

$$= 8\cos 4t + 2t^2$$ **Answer.**

Problem 2. If $\overrightarrow{u(t)} = (6t^2)\hat{\imath} - (t)\hat{\jmath} + (3t^2)\hat{k}$, $\overrightarrow{v(t)} = (t)\hat{\imath} + (t^2)\hat{\jmath} + (2t)\hat{k}$

Find $\left[\overrightarrow{u(t)} \times \overrightarrow{v(t)}\right]'$

Solution. Here we will first find cross product between $\overrightarrow{u(t)}$ and $\overrightarrow{v(t)}$

i.e. $\left[\overrightarrow{u(t)} \times \overrightarrow{v(t)}\right] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6t^2 & -t & 3t^2 \\ t & t^2 & 2t \end{vmatrix} = -(2t^2 + 3t^4)\hat{\imath} - 9t^3\hat{\jmath} + (6t^4 + t^2)\hat{k}$

So, $\left[\overrightarrow{u(t)} \times \overrightarrow{v(t)}\right]' = \frac{d}{dt}\left[\overrightarrow{u(t)} \times \overrightarrow{v(t)}\right] = \frac{d}{dt}\left[-(2t^2 + 3t^4)\hat{\imath} - 9t^3\hat{\jmath} + (6t^4 + t^2)\hat{k}\right]$

$$= -(4t + 12t^3)\hat{\imath} - 27t^2\hat{\jmath} + (24t^3 + 2t)\hat{k}$$ **Answer.**

Problem 3. Prove that $\left[\overrightarrow{v(t)} \times \overrightarrow{v'(t)}\right]' = \overrightarrow{v(t)} \times \overrightarrow{v''(t)}$

Solution. Here $\left[\overrightarrow{v(t)} \times \overrightarrow{v'(t)}\right]' = \frac{d}{dt}\left[\overrightarrow{v(t)} \times \overrightarrow{v'(t)}\right]$

$= \overrightarrow{v(t)} \times \frac{d}{dt}\left(\overrightarrow{v'(t)}\right) + \overrightarrow{v'(t)} \times \frac{d}{dt}\left(\overrightarrow{v(t)}\right)$

$= \left[\overrightarrow{v(t)} \times \overrightarrow{v''(t)}\right] + \left[\overrightarrow{v'(t)} \times \overrightarrow{v'(t)}\right]$

$= \left[\overrightarrow{v(t)} \times \overrightarrow{v''(t)}\right] + 0$ (Cross product of a vector with itself is always zero)

$= \overrightarrow{v(t)} \times \overrightarrow{v''(t)}$

Hence proved.

Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (1 + 3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$$

Solution In that example, we found

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k}$$

and

$$|\mathbf{v}| = \sqrt{9 + 4t^2}.$$

Thus,

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3\sin t}{\sqrt{9 + 4t^2}}\mathbf{i} + \frac{3\cos t}{\sqrt{9 + 4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9 + 4t^2}}\mathbf{k}.$$

Question: *A particle moves along the curve* $\bar{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$, *where t is the time. Find the magnitude of the tangential components of its acceleration at* $t = 2$.

Solution., $\vec{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$

$$\text{Velocity} = \frac{d\vec{r}}{dt} = (3t^2 - 4)\hat{i} + (2t + 4)\hat{j} + (16t - 9t^2)\hat{k}$$

$$t = 2, \qquad \text{Velocity} = 8\hat{i} + 8\hat{j} - 4\hat{k}$$

$$\text{Acceleration} = \vec{a} = \frac{d^2\vec{r}}{dt^2} = 6t\,\hat{i} + 2\hat{j} + (16 - 18t)\hat{k}$$

$$t = 2 \qquad \vec{a} = 12\hat{i} + 2\hat{j} - 20\hat{k}$$

The direction of velocity is along tangent.

So the tangent vector is velocity.

Unit tangent vector, $\hat{T} = \frac{\vec{v}}{|v|} = \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{\sqrt{64+64+16}} = \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{12} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$

Tangential component of acceleration, $a_t = \bar{a}.\hat{T}$

$$= (12\hat{i} + 2\hat{j} - 20\hat{k}).\frac{2\hat{i} + 2\hat{j} - \hat{k}}{3} = \frac{24+4+20}{3} = \frac{48}{3} = 16$$